(D1) **Binary Pulsar**

Through systematic searches during the past decades, astronomers have found a large number of millisecond pulsars (spin period $< 10\,\mathrm{ms}$). Majority of these pulsars are found in binaries, with nearly circular orbits.

For a pulsar in a binary orbit, the measured pulsar spin period (P) and the measured line-of-sight acceleration (a) both vary systematically due to orbital motion. For circular orbits, this variation can be described mathematically in terms of orbital phase ϕ ($0 \le \phi \le 2\pi$) as,

$$P(\phi) = P_0 + P_\mathrm{t}\cos\phi \qquad \text{where } P_\mathrm{t} = \frac{2\pi P_0 r}{c P_\mathrm{B}}$$

$$a(\phi) = -a_\mathrm{t}\sin\phi \qquad \text{where } a_\mathrm{t} = \frac{4\pi^2 r}{P_\mathrm{B}^2}$$

where P_B is the orbital period of the binary, P_0 is the intrinsic spin period of the pulsar and r is the radius of the orbit.

The following table gives one such set of measurements of P and a at different heliocentric epochs, T, expressed in truncated Modified Julian Days (tMJD), i.e. number of days since MJD = 2,440,000.

No.	T (tMJD)	P (µs)	a ($\mathrm{m\,s^{-2}}$)
1	5740.654	7587.8889	-0.92 ± 0.08
2	5740.703	7587.8334	-0.24 ± 0.08
3	5746.100	7588.4100	-1.68 ± 0.04
4	5746.675	7588.5810	$+1.67 \pm 0.06$
5	5981.811	7587.8836	$+0.72 \pm 0.06$
6	5983.932	7587.8552	-0.44 ± 0.08
7	6005.893	7589.1029	$+0.52 \pm 0.08$
8	6040.857	7589.1350	$+0.00 \pm 0.04$
9	6335.904	7589.1358	$+0.00 \pm 0.02$

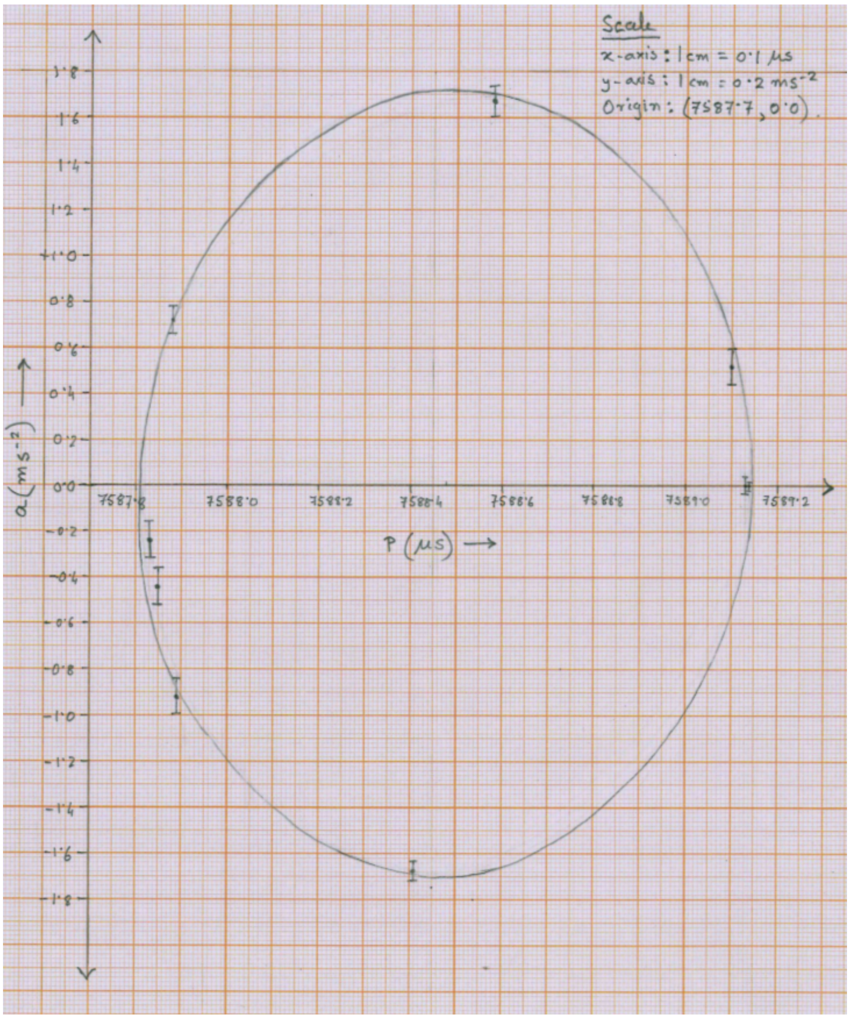
By plotting $a(\phi)$ as a function of $P(\phi)$, we can obtain a parametric curve. As evident from the relations above, this curve in the period-acceleration plane is an ellipse.

In this problem, we estimate the intrinsic spin period, P_0, the orbital period, P_B, and the orbital radius, r, by an analysis of this data set, assuming a circular orbit.

(D1.1) Plot the data, including error bars, in the period-acceleration plane (mark your graph as "D1.1"). **7**

Solution:

Graph Number : D1.1



- **Plot uses more than 50% of graph paper: 0.5**
- **Axes labels (P and a): 0.5**
- **Dimensions of axes: 0.5**
- **Ticks and values on axes (or scale written explicitly): 0.5**
- **Points correctly plotted:**

Points plotted	9	8	7	6	5	< 5
Marks given	4.0	3.5	3.0	2.0	1.0	0

Correctness of points: deduction of 0.5 for each wrong point.

- **Errorbars on points (at least 5): 1.0**

(D1.2) Draw an ellipse that appears to be a best fit to the data (on the same graph "D1.1"). **2**

Solution:
See above

- **Elliptical curve with visual best fit: 1.0**
- **Curve symmetric about $a = 0$ line: 0.5**

- **Curve symmetric about some value of P ($P \approx 7588.48$): 0.5**

(D1.3) From the plot, estimate P_0, P_t and a_t, including error margins. **7**

Solution:
Values are determined from lengths of axes of ellipse and mid-point of P-axis.
Error margins may be determined by estimating extreme ellipses covering the points with errorbars. Any reasonable method of estimating error margins will be accepted.

$2P_t = (1.34 \pm 0.04)\,\mu s$ [(13.4 ± 0.4) cm on graph]

$\boxed{P_t = (0.67 \pm 0.02)\,\mu s}$ **2.0**

$\boxed{P_0 = (7588.48 \pm 0.02)\,\mu s}$ **2.0**

$2a_t = (3.42 \pm 0.12)\,m\,s^{-2}$ [(17.1 ± 0.6) cm on graph] **3.0**

$\boxed{a_t = (1.71 \pm 0.06)\,m\,s^{-2}}$

- **Marking table:**

Parameter	Half credit	Full credit		Half credit
	Minimum	Minimum	Maximum	Maximum
$P_t(\mu s)$	**0.59**	**0.63**	**0.71**	**0.75**
$\delta P_t(\mu s)$	**0.01**	**0.02**	**0.04**	**0.05**
$P_0(\mu s)$	**7588.38**	**7588.43**	**7588.53**	**7588.58**
$\delta P_0(\mu s)$	**0.01**	**0.02**	**0.04**	**0.05**
$a_t(m\,s^{-2})$	**1.61**	**1.65**	**1.77**	**1.81**
$\delta a_t(m\,s^{-2})$	**0.04**	**0.05**	**0.07**	**0.08**

- **Wrong values due to wrong/poor plot/fit in (D1.1) and (D1.2) WILL BE penalised.**
- **Error estimation is based on graph drawing. Quoted values correspond to the envelope of possible ellipses drawn to include all points with errorbars. Any reasonable method to estimate error to be given credit.**

(D1.4) Write expressions for P_B and r in terms of P_0, P_t, a_t. **4**

Solution:
We can easily recover the orbital period (P_B) and the radius of the orbit (r) in a circular orbit:

$$a_t = \frac{4\pi^2}{P_B^2} r$$

$$\therefore r = \frac{P_B^2 a_t}{4\pi^2}$$ **1.0**

$$P_t = \frac{2\pi P_0}{P_B} \times \frac{r}{c}$$

$$= \frac{2\pi P_0}{P_B c} \times \frac{P_B^2 a_t}{4\pi^2} = \frac{P_0 P_B a_t}{2\pi c}$$

$$\therefore \boxed{P_{\rm B} = \frac{P_{\rm t}}{P_0}\frac{2\pi c}{a_{\rm t}}} \quad \mathbf{1.0}$$

$$r = \frac{a_{\rm t}}{4\pi^2}\left(\frac{P_{\rm t}}{P_0}\frac{2\pi c}{a_{\rm t}}\right)^2 \quad \mathbf{1.0}$$

$$\therefore \boxed{r = \left(\frac{P_{\rm t}}{P_0}\right)^2 \frac{c^2}{a_{\rm t}}} \quad \mathbf{1.0}$$

Alternative algebraic routes accepted. Each of $P_{\rm B}$ and r carry 2.0 marks.

(D1.5) Calculate approximate value of $P_{\rm B}$ and r based on your estimations made in (D1.3), including error margins. **6**

Solution:

$$P_{\rm B} = \frac{P_{\rm t}}{P_0}\frac{2\pi c}{a_{\rm t}}$$

$$= \frac{0.67}{7588.48} \times \frac{2\pi \times 2.998 \times 10^8}{1.71}{\rm s}$$

$$= 96\,260\,{\rm s} = 1.125\,70\,{\rm d} \quad \mathbf{1.0}$$

$$\Delta P_{\rm B} = P_{\rm B}\sqrt{\left(\frac{\Delta P_{\rm t}}{P_{\rm t}}\right)^2 + \left(\frac{\Delta P_0}{P_0}\right)^2 + \left(\frac{\Delta a_{\rm t}}{a_{\rm t}}\right)^2} \quad \mathbf{1.0}$$

$$= 1.12570 \times \sqrt{\left(\frac{0.02}{0.67}\right)^2 + \left(\frac{0.02}{7588.48}\right)^2 + \left(\frac{0.06}{1.71}\right)^2}\,{\rm d}$$

$$= 1.12570 \times 0.0461 \simeq 0.052\,{\rm d}$$

$$\therefore \boxed{P_{\rm B} = (1.13 \pm 0.05)\,{\rm d}} \quad \mathbf{1.0}$$

$$r = \left(\frac{P_{\rm t}}{P_0}\right)^2 \frac{c^2}{a_{\rm t}}$$

$$= \left(\frac{0.67}{7588.48}\right)^2 \times \frac{(2.998 \times 10^8)^2}{1.71}{\rm m}$$

$$\therefore r = 4.097\,39 \times 10^8\,{\rm m} = 2.738\,90 \times 10^{-3}\,{\rm AU} \quad \mathbf{1.0}$$

$$\Delta r = r\sqrt{\left(\frac{2\Delta P_{\rm t}}{P_{\rm t}}\right)^2 + \left(\frac{2\Delta P_0}{P_0}\right)^2 + \left(\frac{\Delta a_{\rm t}}{a_{\rm t}}\right)^2} \quad \mathbf{1.0}$$

$$= 2.738\,90 \times 10^{-3} \times \sqrt{\left(\frac{2 \times 0.02}{0.67}\right)^2 + \left(\frac{2 \times 0.02}{7588.48}\right)^2 + \left(\frac{0.06}{1.71}\right)^2}\,{\rm AU}$$

$$= 2.738\,90 \times 10^{-3} \times 0.069\,25\,{\rm AU} \simeq 0.19 \times 10^{-3}\,{\rm AU}$$

$$\boxed{r = (2.74 \pm 0.19) \times 10^{-3}\,{\rm AU}} \quad \mathbf{1.0}$$

Errors in $P_{\rm B}$ and r can be also estimated as maximum possible (worst case) error. In such case, errors would be about 1.5 times the standard error calculated above ($\delta P_{\rm B} = 0.07, \delta r = 0.25$).

(D1.6) Calculate orbital phase, ϕ, corresponding to the epochs of the following five observations **4** in the above table: data rows 1, 4, 6, 8, 9.

Solution:
Using these newly determined orbital parameters, we can calculate the angular orbital phase for each data point, i.e., for each pair of acceleration and period measured (P, a).

$$\phi = \tan^{-1}\left(-\frac{a}{a_t}\frac{P_t}{P - P_0}\right)$$

Care has to be taken to choose the value of the phase from among $\phi, \pi \pm \phi, 2\pi - \phi$, depending on the sign of $\cos\phi$ and $\sin\phi$.

Sr. no.	T (tMJD)	P (µs)	a ($\mathrm{m\,s^{-2}}$)	ϕ
1	5740.654	7587.8889	−0.92	148.62°
4	5746.675	7588.5810	+1.67	278.77°
6	5983.932	7587.8552	−0.44	164.57°
8	6040.857	7589.1350	+0.00	0.00°
9	6335.904	7589.1358	+0.00	0.00°

- **Credit for each correct value: 1.0 for first three, 0.5 for last two.**
- **Credit for $\pi \pm \phi$ or $2\pi - \phi$ is 0.5 per value.**
- **All values wrong due to wrong expression for ϕ gets a maximum of 1.0 mark.**
- **Values in radians accepted.**

(D1.7) Refine the estimate of the orbital period, P_B, using the results in part (D1.6) in the following way:

(D1.7a) First determine the initial epoch, T_0, which corresponds to the nearest epoch of zero phase before the first observation. **2**

Solution:

$$\frac{T_1 - T_0}{P_B} = \frac{\phi_1}{2\pi} \Rightarrow T_0 = T_1 - \frac{\phi_1}{2\pi}P_B$$ **1.0**

$$T_0 = 5740.654 - \frac{148.62°}{360°} \times 1.12570\,\mathrm{tMJD}$$

$$\boxed{T_0 = 5740.189\,\mathrm{tMJD}}$$ **1.0**

Tolerance: ± 0.002 tMJD. Using P_0 instead of P_B gets zero.

(D1.7b) The expected time, T_{calc}, of the estimated phase of each observation is given by **7**

$$T_{calc} = T_0 + \left(n + \frac{\phi}{360°}\right)P_B,$$

where n is the number of full cycle of orbital phases elapsed between T_0 and T_{calc}. Estimate n and T_{calc} for each of the five observations in part (D1.6). Note down difference T_{O-C} between observed T and T_{calc}. Enter these calculations in the table given in the Summary Answersheet.

Solution:

$$T_{\text{calc}} = T_0 + \left(n + \frac{\phi}{360°}\right) P_{\text{B}}$$

where n = Integer part of $[(T - T_0)/P_{\text{B}}]$.

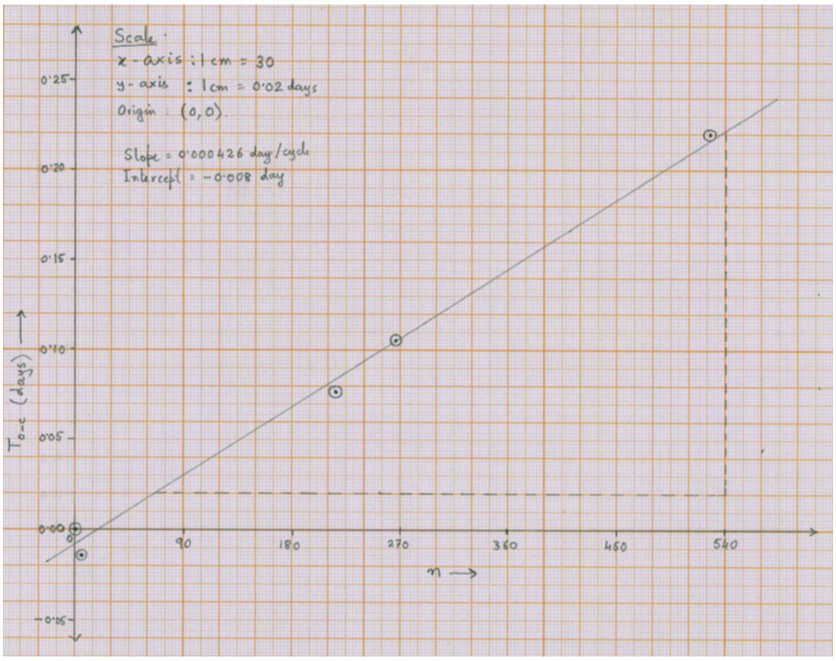
Sr. No.	T (tMJD)	ϕ	n	T_{calc} (MJD)	$T_{\text{O-C}}$ (days)
1	5740.654	148.62°	0	5740.654	0.000
4	5746.675	278.77°	5	5746.689	−0.014
6	5983.932	164.57°	216	5983.855	0.077
8	6040.857	0.00°	267	6040.751	0.106
9	6335.904	0.00°	529	6335.684	0.220

Deduction for each wrong/missing value of n, T_{calc} and $T_{\text{O-C}}$: 0.5 No double penalty in one row.

(D1.7c) Plot $T_{\text{O-C}}$ against n (mark your graph as "D1.7"). **4**

Solution:

Graph Number : D1.7



- **Plot uses more than 50% of graph paper: 0.5**
- **Axes labels ($T_{\text{O-C}}$ and n) including dimensions: 0.5**
- **Ticks and values on axes (or scale written explicitly): 0.5**
- **Points correctly plotted: 0.5 for each point**
- **Goodness of linear fit credited in next part**

(D1.7d) Determine the refined values of the initial epoch, $T_{0,\text{r}}$, and the orbital period, $P_{\text{B,r}}$. **7**

Solution:
A linear fit to the plot of $T_{\mathrm{O-C}}$ vs n gives the offset of period per cycle (slope) and the shift in the zero-phase point (intercept). **2.0**
This concept, which may be evident in the subsequent calculation, gains the credit, explicit statement is not necessary.
From a linear fit,

$$\text{Slope} = 0.000\,43\,\mathrm{d/n} \qquad \text{Intercept} = -0.010\,\mathrm{d}$$ **3.0**

- **Credit for good visual linear fit: 1.0**
- **Correct values of slope and intercept: 1.0 each**
- **Tolerance:** ± 0.00002 **in slope and** ± 0.002 **in intercept.**

$$T_{0,\mathrm{r}} = 5740.189 - 0.010 = \boxed{5740.179\,\mathrm{tMJD}}$$ **1.0**

$$P_{\mathrm{B,r}} = (1.12570 + 0.00043)\,\mathrm{d} = 1.126\,13\,\mathrm{d}$$

$$\boxed{P_{\mathrm{B}} = 1.1261\,\mathrm{d}}$$ **1.0**

Incorrect sign of correction applied carries penalty of 0.5 for each quantity.

(D2) **Distance to the Moon**

Geocentric ephemerides of the Moon for September 2015 are given in the form of a table. Each reading was taken at 00:00 UT.

Date	R.A. (α)			Dec. (δ)			Angular Size (θ)	Phase (ϕ)	Elongation of Moon
	h	m	s	°	′	″	″		
Sep 01	0	36	46.02	3	6	16.8	1991.2	0.927	148.6° W
Sep 02	1	33	51.34	7	32	26.1	1974.0	0.852	134.7° W
Sep 03	2	30	45.03	11	25	31.1	1950.7	0.759	121.1° W
Sep 04	3	27	28.48	14	32	4.3	1923.9	0.655	107.9° W
Sep 05	4	23	52.28	16	43	18.2	1896.3	0.546	95.2° W
Sep 06	5	19	37.25	17	55	4.4	1869.8	0.438	82.8° W
Sep 07	6	14	19.23	18	7	26.6	1845.5	0.336	70.7° W
Sep 08	7	7	35.58	17	23	55.6	1824.3	0.243	59.0° W
Sep 09	7	59	11.04	15	50	33.0	1806.5	0.163	47.5° W
Sep 10	8	49	0.93	13	34	55.6	1792.0	0.097	36.2° W
Sep 11	9	37	11.42	10	45	27.7	1780.6	0.047	25.1° W
Sep 12	10	23	57.77	7	30	47.7	1772.2	0.015	14.1° W
Sep 13	11	9	41.86	3	59	28.8	1766.5	0.001	3.3° W
Sep 14	11	54	49.80	0	19	50.2	1763.7	0.005	7.8° E
Sep 15	12	39	50.01	-3	20	3.7	1763.8	0.026	18.6° E
Sep 16	13	25	11.64	-6	52	18.8	1767.0	0.065	29.5° E
Sep 17	14	11	23.13	-10	9	4.4	1773.8	0.120	40.4° E
Sep 18	14	58	50.47	-13	2	24.7	1784.6	0.189	51.4° E
Sep 19	15	47	54.94	-15	24	14.6	1799.6	0.270	62.5° E
Sep 20	16	38	50.31	-17	6	22.8	1819.1	0.363	73.9° E
Sep 21	17	31	40.04	-18	0	52.3	1843.0	0.463	85.6° E
Sep 22	18	26	15.63	-18	0	41.7	1870.6	0.567	97.6° E
Sep 23	19	22	17.51	-17	0	50.6	1900.9	0.672	110.0° E
Sep 24	20	19	19.45	-14	59	38.0	1931.9	0.772	122.8° E
Sep 25	21	16	55.43	-11	59	59.6	1961.1	0.861	136.2° E
Sep 26	22	14	46.33	-8	10	18.3	1985.5	0.933	150.0° E
Sep 27	23	12	43.63	-3	44	28.7	2002.0	0.981	164.0° E
Sep 28	0	10	48.32	0	58	58.2	2008.3	1.000	178.3° E
Sep 29	1	9	5.89	5	38	54.3	2003.6	0.988	167.4° W
Sep 30	2	7	39.02	9	54	16.1	1988.4	0.947	153.2° W

The composite graphic[1] below shows multiple snapshots of the Moon taken at different times during the total lunar eclipse, which occurred in this month. For each shot, the centre of frame was coinciding with the central north-south line of umbra.

For this problem, assume that the observer is at the centre of the Earth and angular size refers to angular diameter of the relevant object / shadow.

(D2.1) In September 2015, apogee of the lunar orbit is closest to New Moon / First Quarter / Full Moon / Third Quarter. Tick the correct answer in the Summary Answersheet. No justification for your answer is necessary. **3**

Solution:
From the table we see that the angular size of Moon is smallest close to the New Moon day. Thus, the answer is New Moon. **3.0**
Justification is NOT necessary for full credit.

(D2.2) In September 2015, the ascending node of lunar orbit with respect to the ecliptic is closest to New Moon / First Quarter / Full Moon / Third Quarter. Tick the correct answer in the Summary Answersheet. No justification for your answer is necessary. **4**

Solution:
As there is an eclipse happening in this month, the lunar nodes are close to Full Moon day and New Moon day. Next we notice that lowest declination of Moon is just 18°. This means that after the New Moon day, the orbit of Moon is above the ecliptic. In other words, the ascending node is near the New Moon. **4.0**
Justification is NOT necessary for full credit.

(D2.3) Estiamte the eccentricity, e, of the lunar orbit from the given data. **4**

Solution:
The largest angular size of the Moon in the ephemerides is 2008.3″ and the smallest angular size is 1763.7″. The distance is inversely proportional to the angular size. Hence ratio of distance at perigee to the distance at apogee is:

$$\text{Ratio} = \frac{r_{\text{perigee}}}{r_{\text{apogee}}} = \frac{\cancel{a_0}}{1+e} \times \frac{1-e}{\cancel{a_0}}$$ **2.0**

[1]Credit: NASA's Scientific Visualization Studio

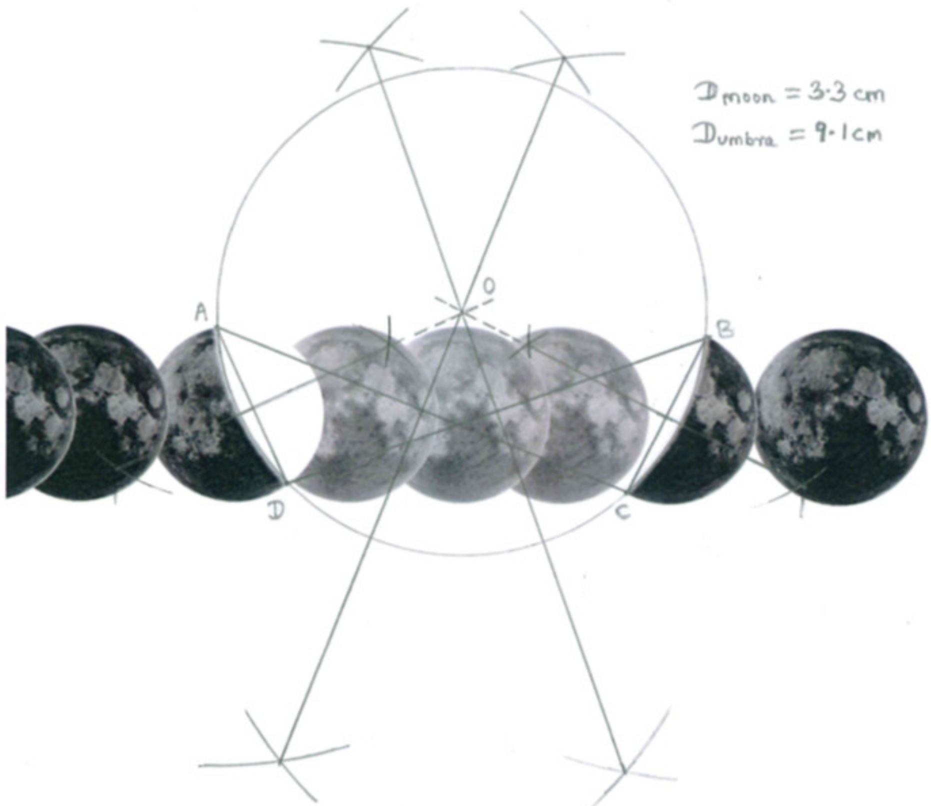
$$\therefore \frac{1-e}{1+e} = \frac{1763.7}{2008.3} = 0.87821$$
$$\therefore e = \frac{1-0.87821}{1+0.87821} = 0.064846$$
$$\boxed{e \simeq 0.065}$$ **2.0**

Rounding off is done to account for the fact that our data are not continuous, hence exact angular sizes at perigee and apogee are not known. A non-rounded answer will also receive full credit.

(D2.4) Estimate the angular size of the umbra, θ_{umbra}, in terms of the angular size of the Moon, θ_{Moon}. Show your working on the image given on the backside of the Summary Answersheet. **8**

Solution:

The following construction is shown. **5.0**



Only two chords are necessary to determine the centre.

The credit is divided in two parts for the drawing:

- **Realisation that centre of umbra circle needs to be determined to find θ_{umbra}: 1.5**
- **Accurate determination of centre of umbra circle by geometric construction: 2.5**
  Determination of centre of hand-drawn circle: maximum 1.5

- **Measuring diameters of umbra and Moon: 0.5 each**

By estimating approximate centre of the shadow in the image, we find out,

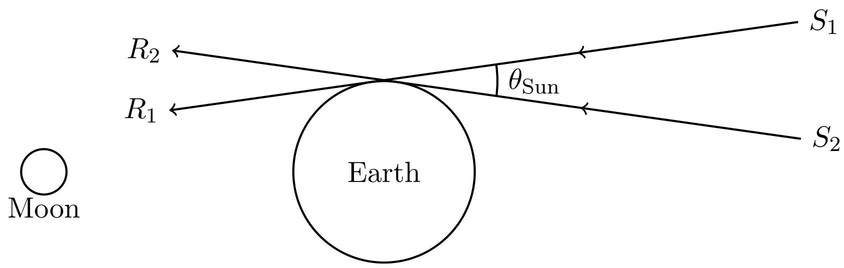
$$\frac{\theta_{\text{umbra}}}{\theta_{\text{Moon}}} = \frac{d_{\text{umbra}}}{d_{\text{Moon}}}$$ **2.0**

$$= \frac{9.1}{3.3} = 2.76$$

$$\therefore \boxed{\theta_{\text{umbra}} \simeq 2.76\theta_{\text{Moon}}}$$ **1.0**

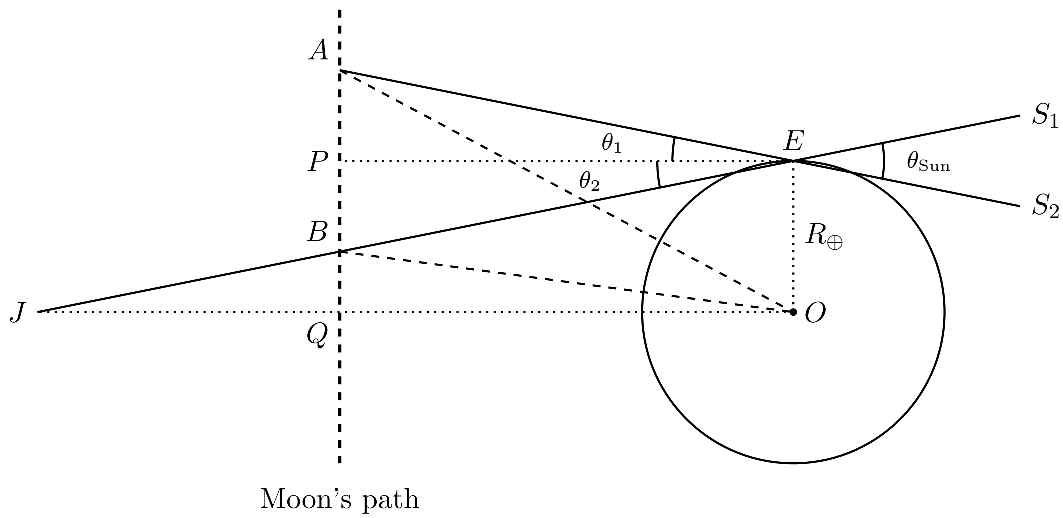
Acceptable range: ± 0.10**.**

(D2.5) The angle subtended by the Sun at Earth on the day of the lunar eclipse is known to be $\theta_{\text{Sun}} = 1915.0''$. In the figure below, S_1R_1 and S_2R_2 are rays coming from diametrically opposite ends of the solar disk. The figure is not to scale. **9**



Calculate the angular size of the penumbra, θ_{penumbra}, in terms of θ_{Moon}. Assume the observer to be at the centre of the Earth.

Solution:
The following diagram needs to be drawn.



Angular size of umbra is $\theta_{\text{umbra}} = 2\measuredangle BOQ$
Angular size of penumbra is $\theta_{\text{penumbra}} = 2\measuredangle AOQ$
We have

$$QA = QP + PA$$
$$= OE + PE \tan\theta_1$$

$$\approx R_\oplus + d_\text{Moon}\theta_1 \qquad \text{since } PA \ll PE$$

$$\approx R_\oplus + d_\text{Moon}\frac{\theta_\text{Sun}}{2} \qquad \text{since } \theta_1 \approx \theta_2 \approx \theta_\text{Sun}/2$$ **2.0**

and

$$\begin{aligned} QB &= QP - PB \\ &= OE - PE\tan\theta_2 \\ &\approx R_\oplus - d_\text{Moon}\theta_2 \\ &\approx R_\oplus - d_\text{Moon}\frac{\theta_\text{Sun}}{2} \end{aligned}$$ **2.0**

$$\therefore \theta_\text{penumbra} = 2\measuredangle AOQ = 2\tan^{-1}\left(\frac{QA}{OQ}\right) \approx 2\frac{QA}{OQ} \qquad \text{since } QA \ll OQ$$

$$= 2\frac{R_\oplus + d_\text{Moon}\frac{\theta_\text{Sun}}{2}}{d_\text{Moon}} = \frac{2R_\oplus}{d_\text{Moon}} + \theta_\text{Sun}$$ **1.0**

and

$$\theta_\text{umbra} = 2\measuredangle BOQ = 2\tan^{-1}\left(\frac{QB}{OQ}\right) \approx 2\frac{QB}{OQ}$$

$$= 2\frac{R_\oplus - d_\text{Moon}\frac{\theta_\text{Sun}}{2}}{d_\text{Moon}} = \frac{2R_\oplus}{d_\text{Moon}} - \theta_\text{Sun}$$ **1.0**

Subtracting,

$$\theta_\text{penumbra} - \theta_\text{umbra} = 2\theta_\text{Sun} \Rightarrow \theta_\text{penumbra} = \theta_\text{umbra} + 2\theta_\text{Sun}$$ **1.0**

We have,

$$\theta_\text{umbra} = 2.76\theta_\text{Moon} \qquad \text{and } \theta_\text{Sun} = 1915.0''$$

From the given data, $\theta_\text{Moon} = 2008.3''$. **1.0**
Therefore,

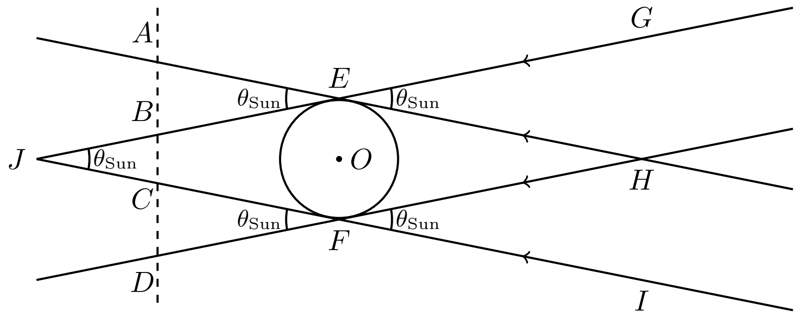
$$\theta_\text{penumbra} = 2.76\theta_\text{Moon} + 2\frac{1915.0}{2008.3}\theta_\text{Moon}$$

$$\boxed{\theta_\text{penumbra} = 4.67\theta_\text{Moon}}$$ **1.0**

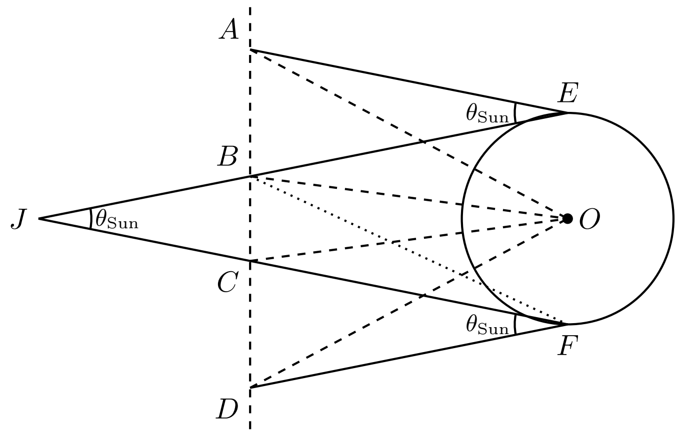
Acceptable range: $4.57\theta_\text{Moon}$ to $4.77\theta_\text{Moon}$.

Alternative solution:

In the figure below, rays HEA and IFC are coming from one edge of solar disk and rays HFD and GEB are coming from the opposite edge. The observer (O) is assumed to be at the centre of the Earth. The Moon travels along the path $ABCD$ during the course of eclipse.



Moon's path



Moon's path

From figure,

$$\measuredangle AEB = \measuredangle GEH = \measuredangle HFI = \measuredangle DFC = \measuredangle EJF = \theta_{\text{Sun}} \quad \mathbf{3.0}$$

$$\theta_{\text{umbra}} = \measuredangle BOC = 2.76\theta_{\text{Moon}} \quad \mathbf{1.0}$$

$$\theta_{\text{penumbra}} = \measuredangle AOD \quad \mathbf{1.0}$$

$$\measuredangle AOD = \measuredangle AOB + \measuredangle BOC + \measuredangle COD$$

$$\measuredangle AOB = \measuredangle AEB \quad \mathbf{1.0}$$

$$\measuredangle COD = \measuredangle CFD \quad \mathbf{1.0}$$

$$\theta_{\text{penumbra}} \simeq \measuredangle AEB + \theta_{\text{umbra}} + \measuredangle CFD$$

$$= 2\theta_{\text{Sun}} + 2.76\theta_{\text{Moon}} = 2 \times 1915.0'' + 2.76 \times 2008.3''$$

$$\boxed{\theta_{\text{penumbra}} = 9372.9'' = 4.67\theta_{\text{Moon}}} \quad \mathbf{2.0}$$

(D2.6) Let θ_{Earth} be angular size of the Earth as seen from the centre of the Moon. Calculate the angular size of the Moon, θ_{Moon}, as would be seen from the centre of the Earth on the eclipse day in terms of θ_{Earth}. **5**

Solution:

From the Moon,

$$\theta_{\text{Earth}} = \frac{2R_{\oplus}}{d_{\text{Moon}}} \quad \mathbf{1.0}$$

From part (D2.5),

$$\theta_{\text{umbra}} + \theta_{\text{penumbra}} = 2\theta_{\text{Earth}}$$ **2.0**

$$\therefore \theta_{\text{Earth}} = \frac{\theta_{\text{umbra}} + \theta_{\text{penumbra}}}{2} = \frac{2.76 + 4.67}{2}\theta_{\text{Moon}} = 3.72\theta_{\text{Moon}}$$

$$\boxed{\theta_{\text{Moon}} = = 0.269\theta_{\text{Earth}}}$$ **2.0**

Alternative solution:

Let us say that the Moon is at position of B. Thus, angular size of Earth as seen from this position will be, (see figure in the previous part)

$$\theta_{\text{Earth}} = \measuredangle EBF = \measuredangle BFD$$ **2.0**

$$= \measuredangle BFC + \measuredangle CFD$$

$$\simeq \theta_{\text{umbra}} + \theta_{\text{Sun}}$$ **1.0**

The angular size of the Full Moon on 28 September as seen in the table is $2008.3''$.

$$\theta_{\text{Earth}} = 2.76 \times 2008.3'' + 1915.0'' = 7453.0''$$

$$\boxed{\theta_{\text{Moon}} = 0.269\theta_{\text{Earth}} = \frac{\theta_{\text{Earth}}}{3.72}}$$ **2.0**

(D2.7) Estimate the radius of the Moon, R_{Moon}, in km from the results above. **3**

Solution:

Thus, the radius of Moon will be,

$$R_{\text{Moon}} = \frac{R_\oplus}{3.72}$$

$$R_{\text{Moon}} = \frac{6371}{3.72}$$ **1.0**

$$\boxed{R_{\text{Moon}} \simeq 1713\,\text{km}}$$ **2.0**

Acceptable range: $\pm 20\,\text{km}$.

(D2.8) Estimate the shortest distance, r_{perigee}, and the farthest distance, r_{apogee}, to the Moon. **4**

Solution:

The shortest and longest distances will be,

$$r_{\text{perigee}} = \frac{2 \times 1713 \times 206265}{2008.3}$$

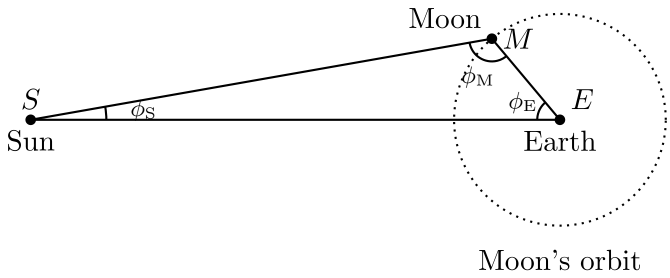
$$\boxed{r_{\text{perigee}} = 3.52 \times 10^5\,\text{km}}$$ **2.0**

$$r_{\text{apogee}} = \frac{2 \times 1713 \times 206265}{1763.7}$$

$$\boxed{r_{\text{apogee}} = 4.01 \times 10^5\,\text{km}}$$ **2.0**

(D2.9) Use appropriate data from September 10 to estimate the distance, d_{Sun}, to the Sun from the Earth. **10**

Solution:



On September 10, phase of Moon is 0.097 and elongation of Moon is 36.2°. Angular size of the Moon on this day is 1792.0″. Therefore, distance to Moon (from Earth) on September 10 is

$$d_{\text{Moon},10} = \frac{2 \times 1713 \times 206265}{1792.0}$$

$$= 3.94 \times 10^5 \text{ km}$$ **2.0**

$$\text{Let } \measuredangle EMS = \phi_M$$

$$\measuredangle ESM = \phi_S$$

$$\measuredangle SEM = \phi_E$$

$$\therefore \phi_E = 36.2^\circ$$ **1.0**

$$\text{phase} = \frac{1 + \cos\phi_M}{2}$$ **2.0**

$$\therefore \phi_M = \cos^{-1}(2 \times \text{ phase } - 1)$$

$$= \cos^{-1}(2 \times 0.097 - 1) = \cos^{-1}(-0.806)$$

$$= 143.71^\circ$$ **2.0**

$$\phi_S = 180^\circ - \phi_E - \phi_M$$

$$= 180^\circ - 36.2^\circ - 143.71^\circ$$

$$= 0.09^\circ$$ **1.0**

Now using sine rule,

$$\frac{d_{\text{Sun}}}{d_{\text{Moon},10}} = \frac{\sin\phi_M}{\sin\phi_S}$$ **2.0**

$$\therefore d_{\text{Sun}} = \frac{3.94 \times 10^8 \times \sin 143.71^\circ}{\sin 0.09^\circ}$$

$$\boxed{d_{\text{Sun}} = 1.48 \times 10^{11} \text{ m}}$$ **1.0**

(D3) **Type IA Supernovae**

Supernovae of type Ia are considered very important for the measurements of large extragalactic distances. The brightening and subsequent dimming of these explosions follow a characteristic light curve, which helps in identifying these as supernovae of type Ia.

Light curves of all type Ia supernovae can be fit to the same model light curve, when they are scaled appropriately. In order to achieve this, we first have to express the light curves in the reference frame of the host galaxy by taking care of the cosmological stretching/dilation of all observed time intervals, $\Delta t_{\rm obs}$, by a factor of $(1+z)$. The time interval in the rest frame of the host galaxy is denoted by $\Delta t_{\rm gal}$.

The rest frame light curve of a supernova changes by two magnitudes compared to the peak in a time interval Δt_0 after the peak. If we further scale the time intervals by a factor of s (i.e. $\Delta t_s = s\Delta t_{\rm gal}$) such that the scaled value of Δt_0 is the same for all supernovae, the light curves turn out to have the same shape. It also turns out that s is related linearly to the absolute magnitude, $M_{\rm peak}$, at the peak luminosity for the supernova. That is, we can write

$$s = a + bM_{\rm peak} ,$$

where a and b are constants. Knowing the scaling factor, one can determine absolute magnitudes of supernovae at unknown distances from the above linear equation.

The table below contains data for three supernovae, including their distance moduli, μ (for the first two), their recession speed, cz, and their apparent magnitudes, $m_{\rm obs}$, at different times. The time $\Delta t_{\rm obs} \equiv t - t_{\rm peak}$ shows number of days from the date at which the respective supernova reached peak brightness. The observed magnitudes have already been corrected for interstellar as well as atmospheric extinction.

Name	SN2006TD	SN2006IS	SN2005LZ
μ (mag)	34.27	35.64	
cz (km s^{-1})	4515	9426	12060
$\Delta t_{\rm obs}$ (days)	$m_{\rm obs}$ (mag)	$m_{\rm obs}$ (mag)	$m_{\rm obs}$ (mag)
−15.00	19.41	18.35	20.18
−10.00	17.48	17.26	18.79
−5.00	16.12	16.42	17.85
0.00	15.74	16.17	17.58
5.00	16.06	16.41	17.72
10.00	16.72	16.82	18.24
15.00	17.53	17.37	18.98
20.00	18.08	17.91	19.62
25.00	18.43	18.39	20.16
30.00	18.64	18.73	20.48

(D3.1) Compute $\Delta t_{\rm gal}$ values for all three supernovae, and fill them in the given blank boxes in the data tables on the BACK side of the Summary Answersheet. On a graph paper, plot the points and draw the three light curves in the rest frame (mark your graph as "D3.1"). **15**

Solution:

Redshifts for the three supernovae are $z_1 = 0.0151$, $z_2 = 0.0314$ and $z_3 = 0.0402$. **1.5**

Filling in the three tables ($\Delta t_{\rm gal}$, third column) **3.5**

SN2006TD			
Δt_{obs}	m_{obs}	Δt_{gal}	Δt_{s}
(d)	(mag)	(d)	(d)
−15.00	19.41	−14.78	−20.00
−10.00	17.48	−9.85	−13.34
−5.00	16.12	−4.93	−6.67
0.00	15.74	0.00	0.00
5.00	16.06	4.93	6.67
10.00	16.72	9.85	13.34
15.00	17.53	14.78	20.00
20.00	18.08	19.70	26.67
25.00	18.43	24.63	33.34
30.00	18.64	29.56	40.01

SN2006IS			
Δt_{obs}	m_{obs}	Δt_{gal}	Δt_{s}
(d)	(mag)	(d)	(d)
−15.00	18.35	−14.54	−14.54
−10.00	17.26	−9.70	−9.70
−5.00	16.42	−4.85	−4.85
0.00	16.17	0.00	0.00
5.00	16.41	4.85	4.85
10.00	16.82	9.70	9.70
15.00	17.37	14.54	14.54
20.00	17.91	19.39	19.39
25.00	18.39	24.24	24.24
30.00	18.73	29.09	29.09

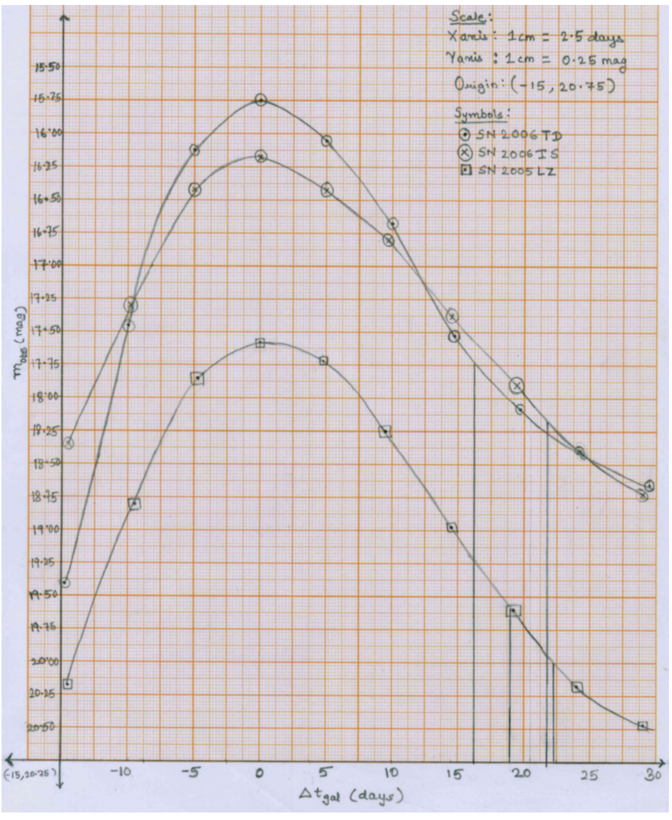
SN2005LZ			
Δt_{obs}	m_{obs}	Δt_{gal}	Δt_{s}
(d)	(mag)	(d)	(d)
−15.00	20.18	−14.42	−17.03
−10.00	18.79	−9.61	−11.35
−5.00	17.85	−4.81	−5.68
0.00	17.58	0.00	0.00
5.00	17.72	4.81	5.68
10.00	18.24	9.61	11.35
15.00	18.98	14.42	17.03
20.00	19.62	19.23	22.70
25.00	20.16	24.03	28.38
30.00	20.48	28.84	34.06

Full marks of 3.5 for all correct values.
Penalty for incorrect values (3×7 independent values):

Incorrect	**1-3**	**4-6**	**7-9**	**10-12**	**13-15**	**16-18**	**19-21**
Deduction	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**

The light curves in galaxy frame would appear as follows **10.0**

Graph Number: D3.1



- **Plot uses more than 50% of graph paper: 0.5**

- **Both axes labels (Δt_{gal} and m_{obs}) present: 0.5**
- **Both dimensions of axes (days and mag) present: 0.5**
- **Ticks and values on axes (or scale written explicitly): 0.5**
- **Points correctly plotted:**
  All points correctly plotted: 5.0
  Penalty for incorrect or missing points:

Incorrect	1	2-4	5-7	8-10	11-13	14-16	17-19	20-22	23-25	26-30
Deduction	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0

- **Smooth curve through points: 1.0 per curve**

(D3.2) Take the scaling factor, s_2, for the supernova SN2006IS to be 1.00. Calculate the scaling factors, s_1 and s_3, for the other two supernovae SN2006TD and SN 2005LZ, respectively, by calculating Δt_0 for them. **5**

Solution:
From the graph D3.1, SN2006IS took 22.0 d to fade by 2 magnitudes.
That is, Δt_0(SN2006IS) = 22.0 d.
Similarly, Δt_0(SN2006TD) = 16.4 d.
And Δt_0(SN2005LZ) = 18.8 d.
Acceptable range: ± 1.0 **days** **3.0**
Thus, stretching factors for these two supernovae are

$$s_1 = \frac{22.2}{16.4} = 1.354$$

$$s_3 = \frac{22.2}{18.8} = 1.181$$ **2.0**

(D3.3) Compute the scaled time differences, Δt_s, for all three supernovae. Write the values for Δt_s in the same data tables on the Summary Answersheet. On another graph paper, plot the points and draw 3 light curves to verify that they now have an identical profile (mark your graph as "D3.3"). **14**

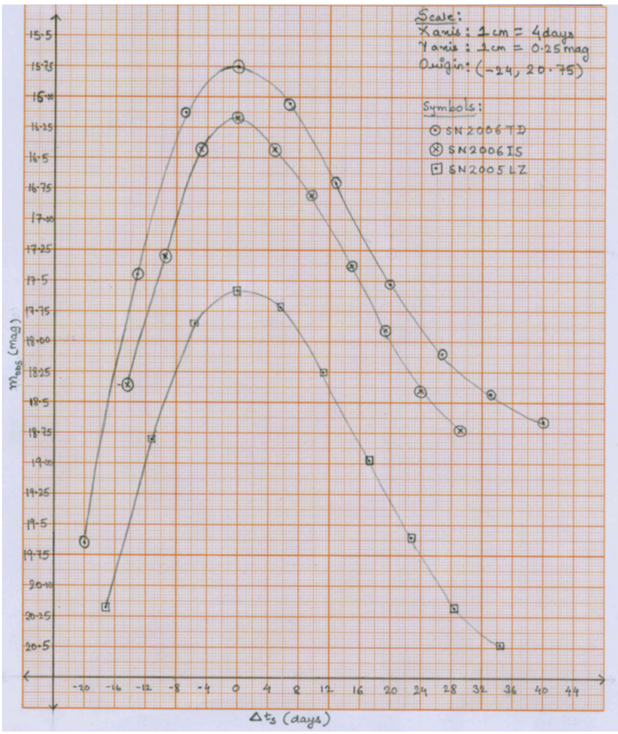
Solution:
Filling the scaled values in the fourth column of the table (Δt_s in table above) **3.5**

Full marks of 3.5 for all correct values.
Penalty for incorrect values (3×7 independent values):

Incorrect	1-3	4-6	7-9	10-12	13-15	16-18	19-21
Deduction	0.5	1.0	1.5	2.0	2.5	3.0	3.5

The scaled light curves would appear as follows,

Graph Number: D3.3



10.5

- **Plot uses more than 50% of graph paper: 0.5**
- **Both axes labels (Δt_s and m_{obs}) present: 0.5**
- **Both dimensions of axes (days and mag) present: 0.5**
- **Ticks and values on axes (or scale written explicitly): 0.5**
- **Points correctly plotted:**
  All points correctly plotted: 5.0
  Penalty for incorrect or missing points:

Incorrect	1	2-4	5-7	8-10	11-13	14-16	17-19	20-22	23-25	26-30
Deduction	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0

- **Smooth curve through points: 1.0 per curve**
- **The curves should show identical profiles.** 0.5

(D3.4) Calculate the absolute magnitudes at peak brightness, $M_{\text{peak},1}$, for SN2006TD and $M_{\text{peak},2}$, for SN2006IS. Use these values to calculate a and b. **6**

Solution:
To get a and b,

$$M_{\text{peak},1} = m_{\text{peak},1} - \mu_1 = 15.74 - 34.27\,\text{mag}$$
$$= -18.53\,\text{mag}$$
$$M_{\text{peak},2} = m_{\text{peak},2} - \mu_2 = 16.17 - 35.64\,\text{mag}$$
$$= -19.47\,\text{mag}$$

2.0

$$\therefore b = \frac{s_1 - s_2}{M_{\text{peak},1} - M_{\text{peak},2}} = \frac{1.354 - 1}{-18.53 - (-19.47)}\,\text{mag}^{-1} = \frac{0.354}{0.94}\,\text{mag}^{-1}$$

$$\boxed{b = 0.3762\,\text{mag}^{-1}}$$ **2.0**

$$a = s_2 - bM_{\text{peak},2} = 1 - 0.3762 \times (-19.47) = 1 + 7.325$$

$$\boxed{a = 8.325}$$ **2.0**

No penalty for missing mag^{-1} **in** b**.**

(D3.5) Calculate the absolute magnitude at peak brightness, $M_{\text{peak},3}$, and distance modulus, μ_3, for SN2005LZ. **[4]**

Solution:

$$s_3 = a + bM_{\text{peak},3}$$

$$\therefore M_{\text{peak},3} = \frac{s_3 - a}{b} = \frac{1.181 - 8.325}{0.3762}\,\text{mag} = \frac{-7.144}{0.3762}\,\text{mag}$$

$$\boxed{M_{\text{peak},3} = -18.99\,\text{mag}}$$ **2.0**

Distance modulus to SN2005LZ is

$$\mu_3 = m_{\text{peak},3} - M_{\text{peak},3} = 17.58 - (-18.99)\,\text{mag}$$

$$\boxed{\mu_3 = 36.57\,\text{mag}}$$ **2.0**

(D3.6) Use the distance modulus μ_3 to estimate the value of Hubble's constant, H_0. Further, estimate the characteristic age of the universe, T_H. **[6]**

Solution:

Distance to SN2005LZ is

$$d_3 = 10^{\left(\frac{\mu_3}{5}+1\right)}\text{pc} = 10^{\left(\frac{\mu_3}{5}+1-6\right)}\text{Mpc}$$

$$= 10^{\left(\frac{36.57}{5}-5\right)}\text{Mpc} = 10^{2.314}\text{Mpc}$$

$$\simeq 206\,\text{Mpc}$$

$$H_0 = \frac{cz_3}{d_3} = \frac{12060}{206}\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$$

$$\boxed{H_0 = 58.5\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}}$$ **4.0**

$$T_\text{H} = \frac{1}{H_0} = \frac{3.086 \times 10^{22}}{58.5 \times 10^3 \times 3.156 \times 10^7}\,\text{yr}$$

$$\boxed{T_\text{H} = 16.7\,\text{Gyr}}$$ **2.0**

Extra factor of $2/3$ **allowed in the value of** T_H**.**